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Cyclic steady thermal response of rapidly switched fixed-bed heat regenerators in counterflow

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Abstract

An analytical solution of the countercurrent thermal regenerator problem when its cyclic steady-state is established is given. It takes account of the flushing phase, i.e. of that part of the blow involved in removing the residue of fluid from a previous blow. In fact, it is possible for some gas particles to remain always within the regenerator, without exiting either end. The proposed analytical approach is based on a Lagrange frame of reference for evaluating the fluid temperature during an entire operation cycle. This temperature is obtained as a function of the time and location within the regenerator and its profile depends on two dimensionless parameters: the reduced length and the flushing ratio. The method allows the matrix temperature profile to be obtained, too. The knowledge of the gas and matrix dimensionless temperature profiles in a graphical form allows a complete estimation of the dimensionless heat exchanged between gas and matrix during a semi-period at a given location. © 1999 Elsevier Science Ltd. All rights reserved.

Nomenclature

A_f	cross-sectional area for flow
c_p, c_v	specific heats at constant pressure and volume
$c_w\rho_w$	heat capacity of matrix per unit of volume
f	friction factor
gz	gas specific potential energy
h	convective heat transfer coefficient
H	specific enthalpy
L	regenerator length
n	nth operation cycle of regenerator
p	pressure
P	matrix wetted perimeter
q	quantity equal to x_i^+
Q	heat exchanged
$\hat{Q}$	heat exchanged per unit of length: $dQ/d\xi$
Q^+	dimensionless heat exchanged: $\hat{Q}/[hP(T_h - T_k)\tau_0]$
r_h	hydraulic radius: A_f/P
R	specific gas constant
s	dimensionless location within the regenerator: x/L
t	time
T	temperature
u	gas flow velocity
U	specific internal energy
x	space coordinate within the regenerator
$\bar{x}_h$, $\bar{x}_k$	lengths defined in the text
x_i	initial location of gas particle
x_i^+	dimensionless initial location of gas particle: x_i/L
y	dimensionless location within the regenerator: ξ/L.

Greek symbols

ε	flushing ratio: $2L/(u\tau_0)$
θ	dimensionless temperature: $(T - T_k)/(T_h - T_k)$
Λ	reduced length: $hL/(c_{p,f}\rho_f\|u\|r_h)$
ξ	generic location within the regenerator
ρ	density
τ	generic time instant
τ_b	blow period time
τ_i	initial time instant
τ_r	reverse period time
τ_0	time of an entire operation cycle: $\tau_b + \tau_r$
τ^+	dimensionless time instant: $(\tau - n\tau_0)/\tau_0$.

Subscripts

b	blow period
f	fluid
h	hot space, inlet of hot fluid
i	initial

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j this index equals either b or r
k cold space, inlet of cold fluid
r reverse period
w matrix.

Superscripts
′ $0 \leqslant x_{\mathrm{i,b}} \leqslant \xi$; $0 \leqslant q_{\mathrm{b}} \leqslant y$
″ $\bar{x}_k \leqslant x_{\mathrm{i,b}} < 0$; $\bar{x}_k/L \leqslant q_{\mathrm{b}} < 0$
* $\xi \leqslant x_{\mathrm{i,r}} \leqslant L$; $y \leqslant q_{\mathrm{r}} \leqslant 1$
** $L < x_{\mathrm{i,r}} \leqslant \bar{x}_{\mathrm{h}}$; $1 < q_{\mathrm{r}} \leqslant \bar{x}_{\mathrm{h}}/L$.

Other symbol
¶ regenerator porosity: void volume, total volume.

1. Introduction

In countercurrent fixed-bed thermal regenerators, the cold and hot fluids pass alternatively and in the opposite direction over a solid wall matrix: the matrix gives up heat when the cold fluid flows over it (heating period or ‘blow period’) and subsequently absorbs this heat from the hot fluid (cooling period or ‘reverse period’), the process being repeated cyclically.

The analysis of a thermal regenerator may be performed following two different approaches [1]: (1) analysis of the start of operation and (2) analysis of the cyclic operation.

If the time required for an element of gas to pass through the regenerator is short compared to the time of either semi-period (blow period or reverse period), then a Euler point of view may be applied to the fluid in both the approaches in order to get the solution of the thermal regenerator problem [1].

If, instead, the time required for a gas particle to pass through the regenerator is approximately equal to the blow time (or reverse time), like it happens in rapidly cycled heat regenerators (e.g., Stirling machine regenerators [2–4] and gas turbine rotary regenerators [5]), then a Lagrange point of view allows the solution to be obtained ‘naturally’ and therefore, in a less complex way, as shown by Organ [6–8].

In addition, in order to account for that part of the blow involved in removing the residue of fluid from a previous blow, Organ introduced a new dimensionless parameter, the flushing ratio ε and the phenomenon was called flushing phase [6, 7]. This parameter was defined as the ratio of the regenerator length to the gas particle excursion amplitude, that is the ratio of the time period required for a gas particle to complete a regenerator traverse to the time of a semi-period.

The concept of flushing ratio for thermal regenerators is very similar to the concept of critical length ratio introduced previously by Organ for the heater and cooler of Stirling cycle machines [9] and later used by Simon et al., in order to characterise, together with other dimensionless parameters, the effect of oscillating flow on pressure drop and heat transfer rates in tubes [10–12].

As a matter of fact, the concept of flushing phase was understood first by Rea and Smith Jr. [13], who introduced a correction factor on the regenerator reduced length Λ introduced by Hausen [1]. However, this factor considered simultaneously the effect of both pressure variations and flushing phase on the performance of thermal regenerators and in addition, its mathematical modelling was not very clear.

The proposed analytical approach does not analyse the transient thermal response of the regenerator until the cyclic operation is established, but directly the final steady periodic state. This state is reached when the heat transferred to the matrix during the flow of the hot gas stream (reverse period) is equal to the heat released from the matrix during the flow of the cold stream (blow period).

The procedure implies firstly the calculation of the fluid temperature as a function of the matrix temperature during both the blow period and the reverse period. This calculation is, in particular, carried out in a closed-form on the basis of some simplifying assumptions (e.g., the thermal capacity of the matrix is assumed infinitely large), by solving the gas energy differential equation along a path followed by a selected gas particle (Lagrange point of view) because of the presence of flushing phase. The next step is to derive, always in an analytical form, the matrix temperature profile by solving a linear integral equation. Once the matrix temperature profile is obtained, the gas temperature may be evaluated and hence presented in a graphical form.

2. Governing equations

The governing differential equations are derived by making a mass balance, a momentum balance and energy balances for the fluid and for the matrix in an elemental control volume, dx long, at position x within the regenerator, as is shown in Fig. 1. The assumptions made in deriving the listed equations are:

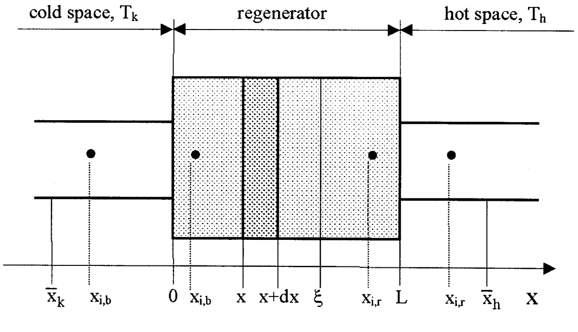

Fig. 1. Schematic representation for the analysis of the regenerator.

(1) the regenerator is one-dimensional, that is no variations normal to the x-direction;
(2) the thermal conductivities of the matrix and fluid are zero parallel to the fluid flow and infinitely large normal to the fluid flow;
(3) the matrix properties specific heat c_w and density ρ_w are independent of temperature;
(4) the cross-sectional free-flow area, A_f, of the regenerator is constant.

For the matrix material within the differential control volume, the basic equation is:

- matrix energy equation:

$$hP[T_f(x,t)-T_w(x,t)] = c_w\rho_w A_f\left(\frac{1-\P}{\P}\right)\frac{\partial T_w(x,t)}{\partial t}. \quad (1)$$

For the fluid within the differential control volume, the basic equations are:

- fluid mass equation:

$$\frac{\partial}{\partial x}(\rho_f u)+\frac{\partial \rho_f}{\partial t}=0 \quad (2)$$

- fluid momentum equation:

$$\rho_f\frac{\partial u}{\partial t}+\rho_f\frac{\partial}{\partial x}\left(\frac{u^2}{2}+gz\right)+\frac{\partial p}{\partial x}+f\rho_f\frac{u^2}{2}\frac{1}{r_h}=0 \quad (3)$$

- fluid energy equation:

$$hP[T_w(x,t)-T_f(x,t)] = A_f\frac{\partial}{\partial t}(\rho_f U_f) + A_f\frac{\partial}{\partial x}\left[\rho_f u\left(H_f+\frac{u^2}{2}+gz\right)\right]. \quad (4)$$

In the listed equations h and f may be linked to u and ρ_f by means of the heat transfer and friction factor correlations given by Gedeon and Wood for oscillating-flow regenerators [14]. The density ρ_f depends on p and T_f by means of the equation of state of the considered fluid.

The following three simplifying assumptions are made:

(1) the specific potential energy of the gas flow is negligible compared to its specific kinetic energy $u^2/2$. Similarly, the specific kinetic energy is negligible compared to the specific enthalpy H_f of the fluid;
(2) the fluid is an ideal gas, whose equation of state is: $p/\rho_f = RT_f$. In addition the caloric equations of state are: $dH_f = c_{p,f}\, dT_f$ and $dU_f = c_{v,f}\, dT_f$;
(3) the specific heats $c_{p,f}$ and $c_{v,f}$ are independent of temperature.

It follows that equations (3) and (4) simplify to

$$\rho_f\frac{\partial u}{\partial t}+\rho_f u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}+f\rho_f\frac{u^2}{2}\frac{1}{r_h}=0 \quad (5)$$

$$hP[T_w(x,t)-T_f(x,t)] = c_{v,f}A_f\frac{\partial}{\partial t}[\rho_f T_f(x,t)] + c_{p,f}A_f\frac{\partial}{\partial x}[\rho_f u T_f(x,t)]. \quad (6)$$

Substituting equation (2) in equation (6) and applying the ideal gas law, with suitable manipulations we get the following expression for the gas energy balance:

$$c_{p,f}\rho_f u\frac{\partial T_f}{\partial x}+c_{p,f}\rho_f\frac{\partial T_f}{\partial t}=\frac{\partial p}{\partial t}-\frac{h}{r_h}(T_f-T_w). \quad (7)$$

Therefore, the set of equations which must be solved simultaneously is represented by four partial differential equations, namely equations (1), (2), (5) and (7), in the unknowns: u, p, T_f and T_w, depending upon x and t. To solve the equations listed before, both the initial and boundary conditions have to be fixed.

For a regenerator working at a cyclic steady-state, the initial conditions do not have any interest. However, it may be noted that certain cyclic steady conditions must be verified at a given position within the regenerator. They are

$$\int_{n\tau_0}^{n\tau_0+\tau_b} h_b(T_w-T_f)\,d\tau = \int_{n\tau_0+\tau_b}^{(n+1)\tau_0} h_r(T_f-T_w)\,d\tau \quad (8)$$

$$c_w\rho_w\oint\frac{\partial T_w}{\partial t}\,dt=0 \quad (9)$$

$$\oint A_f\rho_f u\,dt=0 \quad (10)$$

$$\oint\frac{\partial p}{\partial t}\,dt=0. \quad (11)$$

Equation (8) says that the heat exchanged because of convection between gas and matrix at a fixed location during the blow period must be equal to the heat exchanged at the same location during the reverse period. Equation (9) says that over a complete cycle there can be no net heat storage by the matrix. Equation (10) states that the cyclic change in mass of the gas is zero and equation (11) is merely a statement that the pressure is cyclic. The listed periodic steady conditions must be satisfied by any cyclic steady solution to equations (1), (2), (5) and (7).

As far as the boundary conditions are concerned, at this step it is not convenient to fix them. In fact, they will be fixed after the simplification of the governing equations discussed in the next section.

2.1. Simplification of the governing equations

In order to obtain a closed-form solution of the governing differential equations, further simplifications are made:

(1) the matrix thermal capacity per unit volume $c_w\rho_w$ is infinitely large;

(2) the gas velocity u and pressure p are constant along the regenerator, i.e. $\partial u/\partial x = 0$ and $\partial p/\partial x = 0$;
(3) the flow velocity and the pressure are quasi-stationary, i.e. $\partial u/\partial t = 0$ and $\partial p/\partial t = 0$;
(4) the fluid density ρ_f is constant and evaluated at the average temperature $T_a = (T_h + T_k)/2$;
(5) the coefficient of convective heat transfer h is constant;
(6) the gas flow switches in equal intervals: $\tau_b = \tau_r = \tau_0/2$.

Among the listed simplifications, the former is usually of greater concern. With reference to this, it may be noted that in rapidly cycled heat regenerators (which are herein under discussion) the thermal capacity of the gas during a blow is much smaller than the thermal capacity of the matrix.

Applying the simplifications listed before, the governing differential equations may be rewritten as follows:

- matrix energy balance:

$$\frac{\partial T_w(x,t)}{\partial t} = 0 \Rightarrow T_w = T_w(x). \tag{12}$$

This result implies that the matrix temperature at each location x is essentially constant with time, that is the matrix temperature swing may be neglected;

- gas mass balance: equation (2) is always verified as u and ρ_f have been assumed constant. Therefore, it does not give any useful information;

- gas momentum balance:

$$f\rho_f \frac{u^2}{2}\frac{1}{r_h} = 0 \rightarrow f = 0. \tag{13}$$

This result implies that the fluid is an ideal fluid, that is it is not affected by dynamic viscosity;

- gas energy balance:

$$\frac{\partial}{\partial t} T_{f,j}(x,t) + u_j \frac{\partial}{\partial x} T_{f,j}(x,t) = -\left[\frac{h_j L}{(c_{p,f}\rho_f)_j r_h |u_j|}\right]\left(\frac{|u_j|}{L}\right)[T_{f,j}(x,t) - T_w(x)] \tag{14}$$

where j = b for blow period and j = r for reverse period. According to the space coordinate system labeled in Fig. 1, which considers as positive the gas flow from the cold end of the regenerator to the hot one, it follows $u_b > 0$ and $u_r < 0$. Bearing in mind that the flow velocity has been assumed time-independent, it follows that

$$x = \xi + u_j(t - \tau) \tag{15}$$

in which ξ represents a generic position where an element of gas is located at a generic time instant τ. During the blow period t and τ are in the range $[n\tau_0, n\tau_0 + \tau_0/2]$. During the reverse period t and τ are in the range $[n\tau_0 + \tau_0/2, (n+1)\tau_0]$. If there is not any relevant difference between the blow and reverse periods, like it often approximately corresponds to practical conditions, we have

$$h_b = h_r = h$$
$$u_b = -u_r = u$$
$$(c_{p,f}\rho_f)_b = (c_{p,f}\rho_f)_r = c_{p,f}\rho_f.$$

Therefore: $\Lambda_b = \Lambda_r = \Lambda$. Following Organ's approach [6–8], the first term on the left-side of equation (14) is the substantial derivative, d/dt, of the fluid temperature. Hence, for the fluid energy balance it follows

$$\frac{d}{dt} T_{f,j}[x(t), t] + \frac{\Lambda u}{L} T_{f,j}[x(t), t] = \frac{\Lambda u}{L} T_w[x(t)] \tag{16}$$

where the law of motion of the particle $x(t)$ given by equation (15) may be rewritten as

$$x(t) = \xi \pm u(t - \tau) \tag{17}$$

('+' for blow period and '−' for reverse period). In addition, the cyclic steady conditions (8) and (10) simplify to

$$T_w \tau_0 = \int_{n\tau_0}^{n\tau_0 + \tau_0/2} T_{f,b}\, d\tau + \int_{n\tau_0 + \tau_0/2}^{(n+1)\tau_0} T_{f,r}\, d\tau \tag{18}$$

$$\oint T_f^{-1}\, dt = 0. \tag{19}$$

Equation (18) follows from equation (8) simply applying the result given by equation (12) and bearing in mind that the gas flow has been assumed to switch in equal intervals. Equation (19), instead, derives from equation (10) applying the ideal gas law to ρ_f and bearing in mind that p and u have been assumed time-independent. Equations (9) and (11) are always verified since T_w is independent of time [see equation (12)] and p has been assumed constant.

Finally, the mathematical modelling is represented by only three equations, namely equations (16), (18) and (19), in the unknowns $T_f(x, t)$ and $T_w(x)$. Therefore, equation (19) may be considered redundant.

3. Integration of gas energy equation and initial location of gas particles

To calculate the fluid temperature $T_{f,j}(\xi, \tau)$, the linear first-order differential equation (16) along a path followed by a selected gas particle given by equation (17) has to be solved (Lagrange frame of reference). Its general solution is known [15]:

$$T_{f,j}(\xi,\tau) = T_{f,j}[x(\tau_{i,j}), \tau_{i,j}]\, e^{-\Lambda u(\tau - \tau_{i,j})/L} + \frac{\Lambda u}{L}\int_{\tau_{i,j}}^{\tau} T_w[x(t)]\, e^{\Lambda u(t-\tau)/L}\, dt \tag{20}$$

where $\tau_{i,j}$ represents the initial time instant which corresponds to the beginning of the considered nth semi-period ($\tau_{i,b} = n\tau_0$ and $\tau_{i,r} = n\tau_0 + \tau_0/2$). Therefore, $x(\tau_{i,j})$

appearing in equation (20) represents the initial location $x_{\mathrm{i,j}}$ of the gas particle. It may be evaluated by means of equation (17) simply setting $t = \tau_{\mathrm{i,j}}$:

$$x(\tau_{\mathrm{i,j}}) = x_{\mathrm{i,j}} = \xi \pm u(\tau_{\mathrm{i,j}} - \tau). \tag{21}$$

Therefore, the calculation of the fluid temperature $T_{\mathrm{f,j}}(\xi, \tau)$ given by equation (20) requires the knowledge of the initial fluid temperature $T_{\mathrm{f,j}}(x_{\mathrm{i,j}}, \tau_{\mathrm{i,j}}) = T_{\mathrm{f,j,i}}$ which is strictly linked to the initial location of the gas particle when both the blow and reverse periods start. To define the initial temperature, two further simplifying assumptions are made:

(1) the cold fluid enters the matrix from the left-side at constant temperature T_k, while the hot fluid enters from the right one at constant temperature T_{h} (see Fig. 1);

(2) the temperature of the gas exiting the matrix at the right-hand-side ($x = L$) and entering the hot space becomes instantaneously equal to T_{h}. Similarly, the temperature of the gas exiting the matrix at the left-hand-side ($x = 0$) and entering the cold space becomes instantaneously equal to T_k (see Fig. 1).

3.1. Blow period

Two different locations $x_{\mathrm{i,b}}$ of the fluid particle may be considered at the beginning of the blow period (Fig. 1):

(1) $0 \leqslant x_{\mathrm{i,b}} \leqslant \xi$. The gas particle starting with this location reaches the position ξ at the time $\tau \in [n\tau_0, n\tau_0 + \xi/u]$. In fact, substituting $x_{\mathrm{i,b}} = \xi$ in equation (21) applied to the blow period, it follows that $\tau = n\tau_0$; similarly, setting $x_{\mathrm{i,b}} = 0$ in the same equation, it follows that $\tau = n\tau_0 + \xi/u$. In this case the initial temperature $T_{\mathrm{f,b}}(x_{\mathrm{i,b}}, n\tau_0) = T'_{\mathrm{f,b,i}}$ is not known a priori: it depends on the thermal history of the preceding reverse period. According to the considered initial location $x_{\mathrm{i,b}}$, the fluid temperature given by equation (20) during the blow period becomes

$$T'_{\mathrm{f,b}}(\xi, \tau) = T'_{\mathrm{f,b,i}}\,\mathrm{e}^{-\Lambda u(\tau - n\tau_0)/L} + \frac{\Lambda u}{L}\int_{n\tau_0}^{\tau} T_{\mathrm{w}}[x(t)]\,\mathrm{e}^{\Lambda u(t-\tau)/L}\,\mathrm{d}t \tag{22}$$

(2) $\bar{x}_k \leqslant x_{\mathrm{i,b}} < 0$, where $\bar{x}_k = -(u\tau_0/2 - \xi)$. The gas particle starting with this location reaches the position ξ at the time $\tau \in [n\tau_0 + \xi/u, n\tau_0 + \tau_0/2]$. In this case the initial temperature $T_{\mathrm{f,b}}(x_{\mathrm{i,b}}, n\tau_0) = T''_{\mathrm{f,b,i}}$ is known a priori and is equal to T_k. However, the element of gas does not exchange heat with the matrix between the instant $n\tau_0$ (which corresponds to the location $x_{\mathrm{i,b}}$) and the instant $\hat{\tau}_{\mathrm{b}}$ (which corresponds to the location $x = 0$). Therefore, in equation (20) applied to the blow period the initial instant $\tau_{\mathrm{i,b}} = n\tau_0$ has to be replaced by the instant $\hat{\tau}_{\mathrm{b}}$, as well as the initial position $x_{\mathrm{i,b}}$ has to be replaced by the position $x = 0$. Setting $x = 0$ in equation (17) during the blow period, it is easy to verify that $\hat{\tau}_{\mathrm{b}} = \tau - \xi/u$.

According to the considered initial location $x_{\mathrm{i,b}}$ and bearing in mind that $T_{\mathrm{f,b}}(x = 0, \hat{\tau}_{\mathrm{b}}) = T_k$, the fluid temperature given by equation (20) applied to the blow period is

$$T''_{\mathrm{f,b}}(\xi, \tau) = T_k\,\mathrm{e}^{-\Lambda\xi/L} + \frac{\Lambda u}{L}\int_{\tau - \xi/u}^{\tau} T_{\mathrm{w}}[x(t)]\,\mathrm{e}^{\Lambda u(t-\tau)/L}\,\mathrm{d}t. \tag{23}$$

3.2. Reverse period

Similarly to the blow period, two different locations $x_{\mathrm{i,r}}$ of the fluid particle may be considered when the reverse period starts (Fig. 1):

(1) $\xi \leqslant x_{\mathrm{i,r}} \leqslant L$. The gas element starting with this location reaches the position ξ at the time $\tau \in [n\tau_0 + \tau_0/2, (n\tau_0 + \tau_0/2) + (L - \xi)/u]$. In fact, substituting $x_{\mathrm{i,r}} = \xi$ in equation (21) applied to the reverse period, it follows that $\tau = n\tau_0 + \tau_0/2$; similarly, setting $x_{\mathrm{i,r}} = L$ in the same equation, it follows that $\tau = (n\tau_0 + \tau_0/2) + (L - \xi)/u$. In this case the initial temperature $T_{\mathrm{f,r}}(x_{\mathrm{i,r}}, n\tau_0 + \tau_0/2) = T^*_{\mathrm{f,r,i}}$ is not known a priori: it depends on the thermal history of the preceding blow period. According to the considered initial location $x_{\mathrm{i,r}}$, the fluid temperature given by equation (20) during the reverse period is

$$T^*_{\mathrm{f,r}}(\xi, \tau) = T^*_{\mathrm{f,r,i}}\,\mathrm{e}^{-\Lambda u[\tau - (n\tau_0 + \tau_0/2)]/L} + \frac{\Lambda u}{L}\int_{n\tau_0 + \tau_0/2}^{\tau} T_{\mathrm{w}}[x(t)]\,\mathrm{e}^{\Lambda u(t-\tau)/L}\,\mathrm{d}t \tag{24}$$

(2) $L < x_{\mathrm{i,r}} \leqslant \bar{x}_{\mathrm{h}}$, where $\bar{x}_{\mathrm{h}} = u\tau_0/2 + \xi$. The gas particle starting with this location reaches the position ξ at the time $\tau \in [(n\tau_0 + \tau_0/2) + (L - \xi)/u, (n+1)\tau_0]$. In this case the initial temperature $T_{\mathrm{f,r}}(x_{\mathrm{i,r}}, n\tau_0 + \tau_0/2) = T^{**}_{\mathrm{f,r,i}}$ is known a priori and is equal to T_{h}. Nevertheless, the element of gas does not exchange heat with the matrix between the instant $n\tau_0 + \tau_0/2$ (which corresponds to the location $x_{\mathrm{i,r}}$) and the instant $\hat{\tau}_{\mathrm{r}}$ (which corresponds to the location $x = L$). Therefore, in equation (20) applied to the reverse period the initial instant $\tau_{\mathrm{i,r}} = n\tau_0 + \tau_0/2$ has to be replaced by the instant $\hat{\tau}_{\mathrm{r}}$, as well as the initial position $x_{\mathrm{i,r}}$ has to be replaced by the position $x = L$. Setting $x = L$ in equation (17) applied to the reverse period, it is easy to verify that $\hat{\tau}_{\mathrm{r}} = \tau - (L - \xi)/u$.

According to the considered initial location $x_{\mathrm{i,r}}$ and bearing in mind that $T_{\mathrm{f,r}}(x = L, \hat{\tau}_{\mathrm{r}}) = T_{\mathrm{h}}$, the fluid temperature given by equation (20) during the reverse period becomes

$$T^{**}_{\mathrm{f,r}}(\xi,\tau) = T_{\mathrm{h}}\,\mathrm{e}^{-\Lambda(1-\xi/L)} + \frac{\Lambda u}{L}\int_{\tau-(L-\xi)/u}^{\tau} T_{\mathrm{w}}[x(t)]\,\mathrm{e}^{\Lambda u(t-\tau)/L}\,\mathrm{d}t. \quad (25)$$

4. Calculation of the initial fluid temperatures

To simplify the treatment, the flushing ratio ε has been assumed to be always in the range $\varepsilon \in (0, 1]$. This assumption ensures that there is not a slug of fluid which oscillates within the matrix without exiting either end. In other words, it ensures that:

- the gas particle located at $x_{\mathrm{i,b}} \in [0, \xi]$ comes from the hot space;
- the gas element located at $x_{\mathrm{i,r}} \in [\xi, L]$ comes from the cold space.

4.1. Blow period

The initial temperature $T'_{\mathrm{f,b,i}}$ appearing in equation (22) may be evaluated by means of equation (25), although with some attention. In fact, $T'_{\mathrm{f,b,i}}$ depends on the thermal history of the gas particle in the preceding reverse period whose time interval is $[n\tau_0 - \tau_0/2, n\tau_0]$. Therefore:

$$T'_{\mathrm{f,b,i}} = T_{\mathrm{h}}\,\mathrm{e}^{-\Lambda u(n\tau_0-\hat{\tau}_{\mathrm{b}})/L} + \frac{\Lambda u}{L}\int_{\hat{\tau}_{\mathrm{b}}}^{n\tau_0} T_{\mathrm{w}}[x(t)]\,\mathrm{e}^{\Lambda u(t-n\tau_0)/L}\,\mathrm{d}t \quad (26)$$

where $\tilde{\tau}_{\mathrm{b}}$ may be linked to the generic instant τ of the considered blow period, that is to $\tau \in [n\tau_0, n\tau_0 + \tau_0/2]$, by the relation $\tilde{\tau}_{\mathrm{b}} = 2n\tau_0 - [\tau + (L-\xi)/u]$. The law of motion of the particle $x(t)$, during the reverse period preceding the blow period which is of interest, may be combined to the coordinates ξ and τ by

$$x(t) = \xi - u(\tau - n\tau_0) + u(n\tau_0 - t).$$

4.2. Reverse period

The initial temperature $T^{*}_{\mathrm{f,r,i}}$ appearing in equation (24) may be evaluated by using equation (23), bearing in mind that $T^{*}_{\mathrm{f,r,i}}$ depends on the thermal history of the gas particle in the preceding blow period whose time interval is $[n\tau_0, n\tau_0 + \tau_0/2]$. Therefore:

$$T^{*}_{\mathrm{f,r,i}} = T_k\,\mathrm{e}^{-\Lambda u[(n\tau_0+\tau_0/2)-\hat{\tau}_{\mathrm{r}}]/L} + \frac{\Lambda u}{L}\int_{\tilde{\tau}_{\mathrm{r}}}^{n\tau_0+\tau_0/2} T_{\mathrm{w}}[x(t)]\,\mathrm{e}^{\Lambda u[t-(n\tau_0+\tau_0/2)]/L}\,\mathrm{d}t \quad (27)$$

where $\tilde{\tau}_{\mathrm{r}}$ may be linked to the generic instant τ of the considered reverse period, that it to $\tau \in [n\tau_0 + \tau_0/2, (n+1)\tau_0]$, by the equation $\tilde{\tau}_{\mathrm{r}} = 2(n\tau_0 + \tau_0/2) - (\tau + \xi/u)$. The law of motion of the gas element $x(t)$, during the blow period preceding the reverse period which is herein under discussion, may be combined to the coordinates ξ and τ by

$$x(t) = \xi + u[\tau - (n\tau_0 + \tau_0/2)] - u[(n\tau_0 + \tau_0/2) - t].$$

5. Fluid temperatures as functions of the matrix temperature

Once the initial temperatures $T'_{\mathrm{f,b,i}}$ and $T^{*}_{\mathrm{f,r,i}}$ have been calculated, the gas temperatures $T_{\mathrm{f,b}}$ and $T_{\mathrm{f,r}}$ may be obtained as shown below.

5.1. Blow period

Substituting the temperature $T'_{\mathrm{f,b,i}}$ given by equation (26) in equation (22) and changing the variable of integration in the integrals on the right-hand term of equation (22), time variable $t \to$ space variable x, we get

$$T'_{\mathrm{f,b}}(\xi,\tau) = \mathrm{e}^{-\Lambda\xi/L}\left[T_{\mathrm{h}}\,\mathrm{e}^{\Lambda(2x_{\mathrm{i,b}}/L-1)} - \frac{\Lambda}{L}\mathrm{e}^{2\Lambda x_{\mathrm{i,b}}/L}\int_{L}^{x_{\mathrm{i,b}}} T_{\mathrm{w}}(x)\,\mathrm{e}^{-\Lambda x/L}\,\mathrm{d}x + \frac{\Lambda}{L}\int_{x_{\mathrm{i,b}}}^{\xi} T_{\mathrm{w}}(x)\,\mathrm{e}^{\Lambda x/L}\mathrm{d}x\right] \quad (28)$$

where $T'_{\mathrm{f,b}}$ depends on time τ by means of $x_{\mathrm{i,b}}$, as shown by equation (21) applied to the blow period (j = b and '+'). Instead, changing the variable of integration in the integral on the right-side of equation (23), $t \to x$, we get

$$T''_{\mathrm{f,b}}(\xi) = \mathrm{e}^{-\Lambda\xi/L}\left[T_k + \frac{\Lambda}{L}\int_0^{\xi} T_{\mathrm{w}}(x)\,\mathrm{e}^{\Lambda x/L}\,\mathrm{d}x\right] \quad (29)$$

where $T''_{\mathrm{f,b}}$ is independent of time. Therefore:

$$T_{\mathrm{f,b}} = \begin{cases} T'_{\mathrm{f,b}}(\xi,\tau) & \tau \in [n\tau_0, n\tau_0 + \xi/u] \\ T''_{\mathrm{f,b}}(\xi) & \tau \in [n\tau_0 + \xi/u, n\tau_0 + \tau_0/2] \end{cases}.$$

In a dimensionless form:

$$\theta_{\mathrm{f,b}} = \begin{cases} \theta'_{\mathrm{f,b}}(y,\tau^+) & \tau^+ \in [0, \varepsilon y/2] \\ \theta''_{\mathrm{f,b}}(y) & \tau^+ \in [\varepsilon y/2, 1/2] \end{cases} \quad (30)$$

where $\theta'_{\mathrm{f,b}}$ and $\theta''_{\mathrm{f,b}}$ may be obtained normalising equations (28) and (29), respectively:

$$\theta'_{\mathrm{f,b}}(y,\tau^+) = \mathrm{e}^{-\Lambda y}\left[\mathrm{e}^{\Lambda(2q_{\mathrm{b}}-1)} - \Lambda\,\mathrm{e}^{2\Lambda q_{\mathrm{b}}}\int_1^{q_{\mathrm{b}}} \theta_{\mathrm{w}}(s)\,\mathrm{e}^{-\Lambda s}\,\mathrm{d}s + \Lambda\int_{q_{\mathrm{b}}}^{y} \theta_{\mathrm{w}}(s)\,\mathrm{e}^{\Lambda s}\,\mathrm{d}s\right] \quad (31)$$

$$\theta''_{\mathrm{f,b}}(y) = \Lambda\,\mathrm{e}^{-\Lambda y}\int_0^{y} \theta_{\mathrm{w}}(s)\,\mathrm{e}^{\Lambda s}\,\mathrm{d}s. \quad (32)$$

In equation (31) $q_{\mathrm{b}} = x^{+}_{\mathrm{i,b}} = y - 2\tau^+/\varepsilon$.

5.2. Reverse period

Substituting the temperature $T^*_{\mathrm{f,r,i}}$ given by equation (27) in equation (24) and changing the variable of integration in the integrals on the right-side of equation (24), $t \rightarrow x$, we get

$$T^*_{\mathrm{f,r}}(\xi,\tau) = \mathrm{e}^{\Lambda\xi/L}\left[T_k\,\mathrm{e}^{-2\Lambda x_{\mathrm{i,r}}/L} + \frac{\Lambda}{L}\mathrm{e}^{-2\Lambda x_{\mathrm{i,r}}/L}\int_0^{x_{\mathrm{i,r}}} T_{\mathrm{w}}(x)\,\mathrm{e}^{\Lambda x/L}\,\mathrm{d}x - \frac{\Lambda}{L}\int_{x_{\mathrm{i,r}}}^{\xi} T_{\mathrm{w}}(x)\,\mathrm{e}^{-\Lambda x/L}\,\mathrm{d}x\right] \qquad (33)$$

where $T^*_{\mathrm{f,r}}$ depends on time τ by means of $x_{\mathrm{i,r}}$. Instead, changing the variable integration in the integral on the right-hand term of equation (25), $t \rightarrow x$, we have

$$T^*_{\mathrm{f,r}}{}^*(\xi) = \mathrm{e}^{\Lambda\xi/L}\left[T_{\mathrm{h}}\,\mathrm{e}^{-\Lambda} - \frac{\Lambda}{L}\int_L^{\xi} T_{\mathrm{w}}(x)\,\mathrm{e}^{-\Lambda x/L}\,\mathrm{d}x\right] \qquad (34)$$

in which $T^*_{\mathrm{f,r}}{}^*$ is independent of time. Therefore:

$$T_{\mathrm{f,r}} = \begin{cases} T^*_{\mathrm{f,r}}(\xi,\tau) & \tau\in[n\tau_0+\tau_0/2, (n\tau_0+\tau_0/2)+(L-\xi)/u] \\ T^*_{\mathrm{f,r}}{}^*(\xi) & \tau\in[(n\tau_0+\tau_0/2)+(L-\xi)/u, (n+1)\tau_0] \end{cases}.$$

In a dimensionless form:

$$\theta_{\mathrm{f,r}} = \begin{cases} \theta^*_{\mathrm{f,r}}(y,\tau^+) & \tau^+\in[1/2, 1/2+\varepsilon(1-y)/2] \\ \theta^*_{\mathrm{f,r}}{}^*(y) & \tau^+\in[1/2+\varepsilon(1-y)/2, 1] \end{cases} \qquad (35)$$

where $\theta^*_{\mathrm{f,r}}$ and $\theta^*_{\mathrm{f,r}}{}^*$ may be obtained by normalising equations (33) and (34), respectively:

$$\theta^*_{\mathrm{f,r}}(y,\tau^+) = \Lambda\,\mathrm{e}^{\Lambda y}\left[\mathrm{e}^{-2\Lambda q_{\mathrm{r}}}\int_0^{q_{\mathrm{r}}} \theta_{\mathrm{w}}(s)\,\mathrm{e}^{\Lambda s}\,\mathrm{d}s - \int_{q_{\mathrm{r}}}^{y} \theta_{\mathrm{w}}(s)\,\mathrm{e}^{-\Lambda s}\,\mathrm{d}s\right] \qquad (36)$$

$$\theta^{**}_{\mathrm{f,r}}(y) = \mathrm{e}^{\Lambda y}\left[\mathrm{e}^{-\Lambda} - \Lambda\int_1^{y} \theta_{\mathrm{w}}(s)\,\mathrm{e}^{-\Lambda s}\,\mathrm{d}s\right]. \qquad (37)$$

In equation (36) $q_{\mathrm{r}} = x^+_{\mathrm{i,r}} = y+2(\tau^+-1/2)/\varepsilon$.

6. Matrix temperature

The matrix temperature $\theta_{\mathrm{w}}(s)$, which appears in the integrals of equations (31), (32), (36) and (37), may be evaluated by means of equation (18). This equation can be suitably rewritten in a dimensionless form as follows

$$\theta_{\mathrm{w}} = \int_0^{1/2} \theta_{\mathrm{f,b}}\,\mathrm{d}\tau^+ + \int_{1/2}^{1} \theta_{\mathrm{f,r}}\,\mathrm{d}\tau^+ \qquad (38)$$

whose integrals on the right-hand term, bearing in mind equations (30) and (35), are

$$\int_0^{1/2} \theta_{\mathrm{f,b}}\,\mathrm{d}\tau^+ = \int_0^{\tau^+_{\mathrm{c,b}}} \theta'_{\mathrm{f,b}}\,\mathrm{d}\tau^+ + \int_{\tau^+_{\mathrm{c,b}}}^{1/2} \theta''_{\mathrm{f,b}}\,\mathrm{d}\tau^+ \qquad (39)$$

$$\int_{1/2}^{1} \theta_{\mathrm{f,r}}\,\mathrm{d}\tau^+ = \int_{1/2}^{\tau^+_{\mathrm{c,r}}} \theta^*_{\mathrm{f,r}}\,\mathrm{d}\tau^+ + \int_{\tau^+_{\mathrm{c,r}}}^{1} \theta^{**}_{\mathrm{f,r}}\,\mathrm{d}\tau^+ \qquad (40)$$

where $\tau^+_{\mathrm{c,b}} = \varepsilon y/2$ and $\tau^+_{\mathrm{c,r}} = 1/2+\varepsilon(1-y)/2$. To evaluate the matrix temperature θ_{w}, it is convenient to write it as follows:

$$\theta_{\mathrm{w}}(y) = \theta_{\mathrm{w},\varepsilon=0}(y) + F(y) \qquad (41)$$

where the function $F(y)$ represents the difference between the actual temperature $\theta_{\mathrm{w}}(y)$ when $\varepsilon\in(0,1]$ and the linear temperature $\theta_{\mathrm{w},\varepsilon=0}(y)$ when $\varepsilon = 0$. It is given by [16, 17]

$$\theta_{\mathrm{w},\varepsilon=0}(y) = \frac{\Lambda}{\Lambda+2}y + \frac{1}{\Lambda+2}.$$

Substituting equation (41) in equations (31), (32), (36) and (37), new expressions for the temperatures $\theta'_{\mathrm{f,b}}$, $\theta''_{\mathrm{f,b}}$, $\theta^*_{\mathrm{f,r}}$ and $\theta^{**}_{\mathrm{f,r}}$ may be obtained. They are

$$\theta'_{\mathrm{f,b}} = \frac{\Lambda}{\Lambda+2}y + \frac{2}{\Lambda+2}\mathrm{e}^{\Lambda(q_{\mathrm{b}}-y)} + \Lambda\,\mathrm{e}^{-\Lambda y}\left[\int_{q_{\mathrm{b}}}^{y} F(s)\,\mathrm{e}^{\Lambda s}\,\mathrm{d}s - \mathrm{e}^{2\Lambda q_{\mathrm{b}}}\int_1^{q_{\mathrm{b}}} F(s)\,\mathrm{e}^{-\Lambda s}\,\mathrm{d}s\right] \qquad (42)$$

$$\theta''_{\mathrm{f,b}} = \frac{\Lambda}{\Lambda+2}y + \Lambda\,\mathrm{e}^{-\Lambda y}\int_0^{y} F(s)\,\mathrm{e}^{\Lambda s}\,\mathrm{d}s \qquad (43)$$

$$\theta^*_{\mathrm{f,r}} = \frac{\Lambda}{\Lambda+2}y + \frac{2}{\Lambda+2} - \frac{2}{\Lambda+2}\mathrm{e}^{\Lambda(y-q_{\mathrm{r}})} - \Lambda\,\mathrm{e}^{-\Lambda y} \cdot \left[\mathrm{e}^{-2\Lambda q_{\mathrm{r}}}\int_{q_{\mathrm{r}}}^{0} F(s)\,\mathrm{e}^{\Lambda s}\,\mathrm{d}s + \int_{q_{\mathrm{r}}}^{y} F(s)\,\mathrm{e}^{-\Lambda s}\,\mathrm{d}s\right] \qquad (44)$$

$$\theta^{**}_{\mathrm{f,r}} = \frac{\Lambda}{\Lambda+2}y + \frac{2}{\Lambda+2} + \Lambda\,\mathrm{e}^{-\Lambda y}\int_y^{1} F(s)\,\mathrm{e}^{-\Lambda s}\,\mathrm{d}s. \qquad (45)$$

Now, substituting equations (42)–(45) in equation (38), combined to equations (39) and (40) and applying the integration by parts formula linked to the Leibnitz' theorem for differentiation of an integral, we have

$$F(y) = H(y) + \int_0^1 F(s)K(s,y)\,\mathrm{d}s. \qquad (46)$$

This is a linear integral equation generally known as the Fredholm equation of the second kind [18], in which y is the independent variable, $H(y)$ and $K(s,y)$ are known functions given in Appendix A and $F(y)$ is the unknown function. Applying the integral equation theory [18], it may be proven easily that equation (46) has a solution and it is unique. It is

$$F(y) = (A_1y^2+B_1y+C_1)\,\mathrm{e}^{\Lambda y} + (A_2y^2+B_2y+C_2)\,\mathrm{e}^{-\Lambda y} \qquad (47)$$

where the coefficients A_k, B_k and C_k ($k = 1,2$) are given in Appendix B. Because the function $F(y)$ is not linear along the regenerator (although it is very close to be linear), the dimensionless matrix temperature given by equation (41) is not rigorously linear with the dimensionless space coordinate y.

7. Fluid temperatures

Once the $F(y)$ function has been evaluated, the gas temperatures within the regenerator during both the blow period and the reverse period may be obtained.

7.1. Blow period

Substituting the $F(y)$ function given by equation (47) in equations (42) and (43) and solving the integrals appearing on the right-hand terms of these equations, we get

$$\begin{aligned}\theta'_{\mathrm{f,b}} = {} & \frac{\Lambda}{\Lambda+2}y+\frac{2}{\Lambda+2}\mathrm{e}^{\Lambda(q_{\mathrm{b}}-y)} \\ & +\mathrm{e}^{-\Lambda y}\pi_2(y)+\mathrm{e}^{-\Lambda y}\{\mathrm{e}^{2\Lambda q_{\mathrm{b}}}[\pi_3(y=1) \\ & +(\mu_1 q_{\mathrm{b}}^3+\mu_2 q_{\mathrm{b}}^2+\mu_3 q_{\mathrm{b}})]+(\sigma_1 q_{\mathrm{b}}^3+\sigma_2 q_{\mathrm{b}}^2+\sigma_3 q_{\mathrm{b}})\} \qquad (48)\end{aligned}$$

$$\theta''_{\mathrm{f,b}} = \frac{\Lambda}{\Lambda+2}y+\pi_1(y)\,\mathrm{e}^{-\Lambda y}. \qquad (49)$$

In equation (48) the functions $\pi_k(y)$ ($k = 2, 3$), as well as the constant σ_k and μ_k ($k = 1, 2, 3$), are given in Appendix C. It may be noted that when $\varepsilon = 0$, $\theta'_{\mathrm{f,b}}$ is defined over the domain $\tau^+ \in [0, 0]$. Therefore, it does not exist at any time. In equation (49) the function $\pi_1(y)$ is still given in Appendix C. It may be observed that when $\varepsilon = 0$, $\theta''_{\mathrm{f,b}}$ is defined over the domain $\tau^+ \in [0, 1/2]$ which represents the whole blow period and $\pi_1(y) = 0$ (see Appendices B and C).

The fluid temperature during the blow period, given by equation (30) linked to equations (48) and (49), is represented graphically in Fig. 2 as a function of τ^+ with y as a parameter. The matrix temperature (time-independent) is represented too. It may be noted that the fluid temperature function is not continuous at the point $\tau^+ = \varepsilon y/2$. This discontinuity is due to the second simplifying assumption made in Section 3 and it may also be observed in Fig. 3, where $\theta_{\mathrm{f,b}}$ is plotted against y with τ^+ as a parameter.

Figure 2 shows that, at a fixed location y within the regenerator, the matrix does not give up heat to the gas at all times during the whole blow period. In fact, when $\tau^+ \in [0, \tau^+_{\mathrm{ba}}]$, the fluid gives up heat to the matrix and subsequently, when $\tau^+ \in [\tau^+_{\mathrm{ba}}, 1/2]$, it absorbs heat from the matrix, where τ^+_{ba} is the abscissa value of the point of intersection of curves $\theta_{\mathrm{f,b}}$ and θ_{w}. In particular, $\tau^+_{\mathrm{ba}} = \varepsilon y/2$ in Figs 2(a)–(c).

It may be easily proven that the area enclosed between curves θ_{w} and $\theta_{\mathrm{f,b}}$ in Fig. 2, given by $\int_0^{1/2}(\theta_{\mathrm{w}}-\theta_{\mathrm{f,b}})\,\mathrm{d}\tau^+$, represents the dimensionless heat Q^+_{b} supplied from the matrix to the gas during the blow period at a given location y. This heat is:

$$Q^+_{\mathrm{b}} = \frac{\hat{Q}_{\mathrm{b}}}{2\hat{Q}_{\mathrm{b,max}}} = \frac{\displaystyle\int_{n\tau_0}^{n\tau_0+\tau_0/2} hP(T_{\mathrm{w}}-T_{\mathrm{f,b}})\,\mathrm{d}\tau}{2hP(T_{\mathrm{h}}-T_k)(\tau_0/2)}$$

and may be defined as the ratio of the heat $\hat{Q}_{\mathrm{b}}$ actually exchanged during the blow period per unit of length to an ideal amount of heat (maximum) $\hat{Q}_{\mathrm{b,max}}$ which would be exchanged during the same period per unit of length if the matrix temperature could be equal to T_{h} and the fluid temperature could be equal to T_k (multiply by 2).

7.2. Reverse period

Substituting the $F(y)$ function given by equation (47) in equations (44) and (45) and solving the integrals appearing on the right-hand terms of these equations, we obtain

$$\begin{aligned}\theta^{*}_{\mathrm{f,r}} = {} & \frac{\Lambda}{\Lambda+2}y+\frac{2}{\Lambda+2}[1-\mathrm{e}^{\Lambda(y-q_{\mathrm{r}})}] \\ & -\mathrm{e}^{\Lambda y}\pi_3(y)-\mathrm{e}^{\Lambda y}\{\mathrm{e}^{-2\Lambda q_{\mathrm{r}}}[\pi_2(y=0) \\ & +(\sigma_1 q_{\mathrm{r}}^3+\sigma_2 q_{\mathrm{r}}^2+\sigma_3 q_{\mathrm{r}})]+(\mu_1 q_{\mathrm{r}}^3+\mu_2 q_{\mathrm{r}}^2+\mu_3 q_{\mathrm{r}})\} \qquad (50)\end{aligned}$$

$$\theta^{**}_{\mathrm{f,r}} = \frac{\Lambda}{\Lambda+2}y+\frac{2}{\Lambda+2}+\mathrm{e}^{\Lambda y}[\pi_3(y=1)-\pi_3(y)]. \qquad (51)$$

In equation (50) the functions π_k ($k = 2, 3$) and the constants σ_k and μ_k ($k = 1, 2, 3$), are given in Appendix C. It may be noted that when $\varepsilon = 0$, $\theta^{*}_{\mathrm{f,r}}$ is defined over the domain $\tau^+ \in [1/2, 1/2]$. Therefore, this function does not exist at any time. In equation (51) the function $\pi_3(y)$ is still given in Appendix C. It may be observed that when $\varepsilon = 0$, $\theta^{**}_{\mathrm{f,r}}$ is defined over the domain $\tau^+ \in [1/2, 1]$ which represents the whole reverse period and $\pi_3(y) = \pi_3(y = 1) = 0$ (see Appendices B and C).

The fluid temperature during the reverse period is represented graphically in Fig. 4 as a function of τ^+ with y as a parameter. The matrix temperature is represented too. A discontinuity in the fluid temperature function when $\tau^+ = 1/2+\varepsilon(1-y)/2$ is present. It may be also noted in Fig. 5, where $\theta_{\mathrm{f,r}}$ is plotted against y with τ^+ as a parameter.

Figure 4 shows that at a fixed location y the gas does not give up heat to the matrix at all times during the entire reverse period. In fact, when $\tau^+ \in [1/2, \tau^+_{\mathrm{ra}}]$, the matrix gives up heat to the fluid and subsequently, when $\tau^+ \in [\tau^+_{\mathrm{ra}}, 1]$, it absorbs heat from the fluid, where τ^+_{ra} is the abscissa value of the point of intersection of curves $\theta_{\mathrm{f,r}}$ and θ_{w}. In particular, $\tau^+_{\mathrm{ra}} = 1/2+\varepsilon(1-y)/2$ in Figs 4(d)–(f).

The area enclosed between curves $\theta_{\mathrm{f,r}}$ and θ_{w} in Fig. 4, given by $\int_{1/2}^{1}(\theta_{\mathrm{f,r}}-\theta_{\mathrm{w}})\,\mathrm{d}\tau^+$, represents the dimensionless

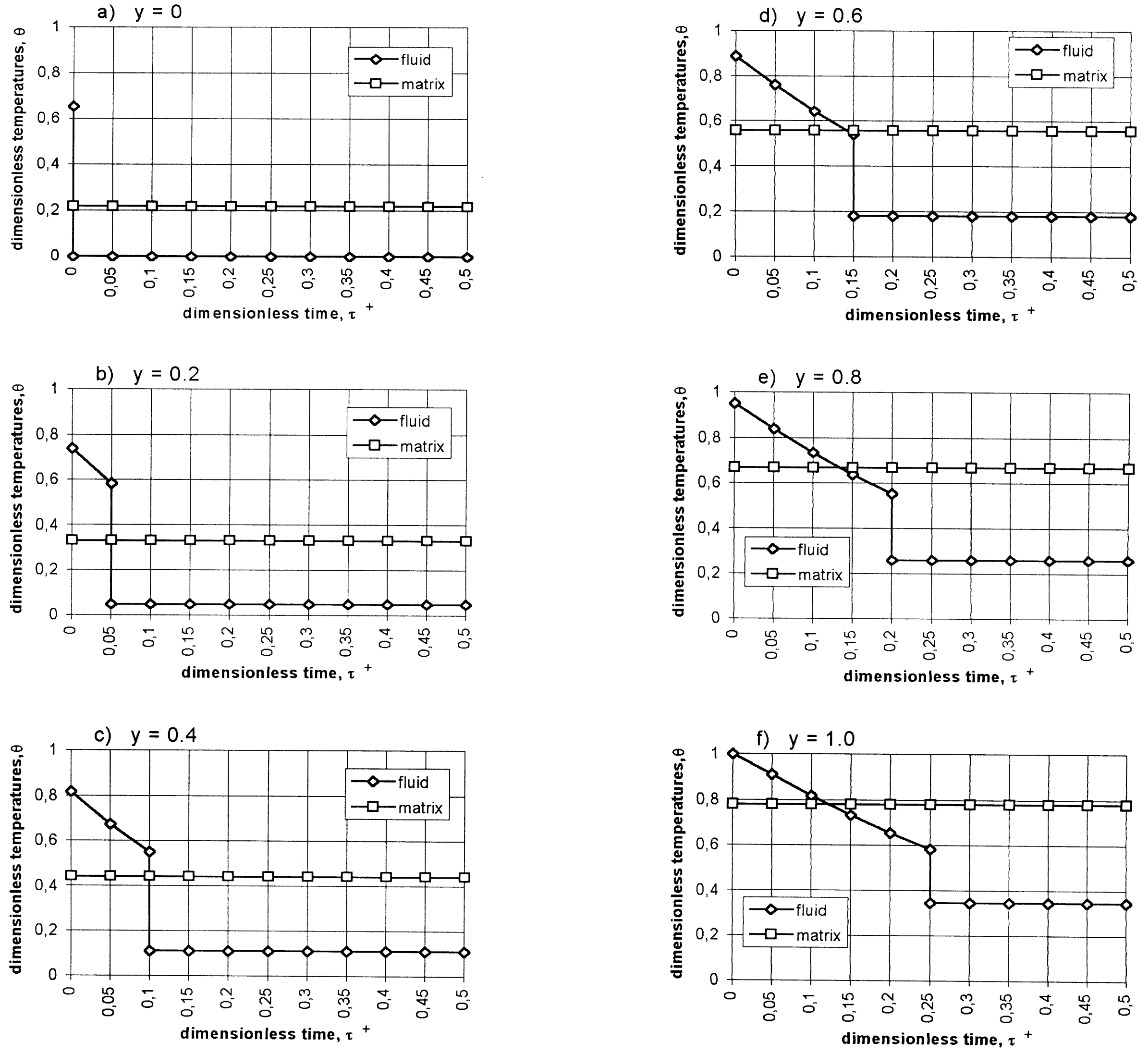

Fig. 2. Dimensionless temperatures (fluid and matrix) as a function of dimensionless time τ^+ with y as a parameter during the blow period. In all cases (a)–(f) $\varepsilon = 0.5$ and $\Lambda = 1$.

heat Q_r^+ supplied from the gas to the matrix during the reverse period at a given location y. This heat is:

$$Q_r^+ = \frac{\hat{Q}_r}{2\hat{Q}_{r,\max}} = \frac{\int_{n\tau_0+\tau_0/2}^{(n+1)\tau_0} hP(T_{f,r}-T_w)\,d\tau}{2hP(T_h-T_k)(\tau_0/2)}$$

where $\hat{Q}_r = \hat{Q}_b$ ($\Rightarrow Q_r^+ = Q_b^+$) because the regenerator is working at a cyclic steady-state.

7.3. *Entire operation cycle*

Combining curves of Figs 2 and 4 yields curves shown in Fig. 6, where the fluid temperature is plotted vs τ^+ with y as a parameter during an entire operation cycle of the regenerator. It may be observed graphically that: (1) the following relationships are verified at any $y \in [0, 1]$:

$$\theta''_{f,b}(y, \tau^+ = 1/2) = \theta^*_{f,r}(y, \tau^+ = 1/2)$$

$$\theta^{**}_{f,r}(y, \tau^+ = 1) = \theta'_{f,b}(y, \tau^+ = 0).$$

It is an elementary, though lengthy matter, to prove analytically the relations listed before; (2) θ_f, and, therefore, T_f, is a periodic and steady function.

8. Conclusions

The proposed fully analytical method allows the cyclic steady operation of a rapidly switched fixed-bed heat

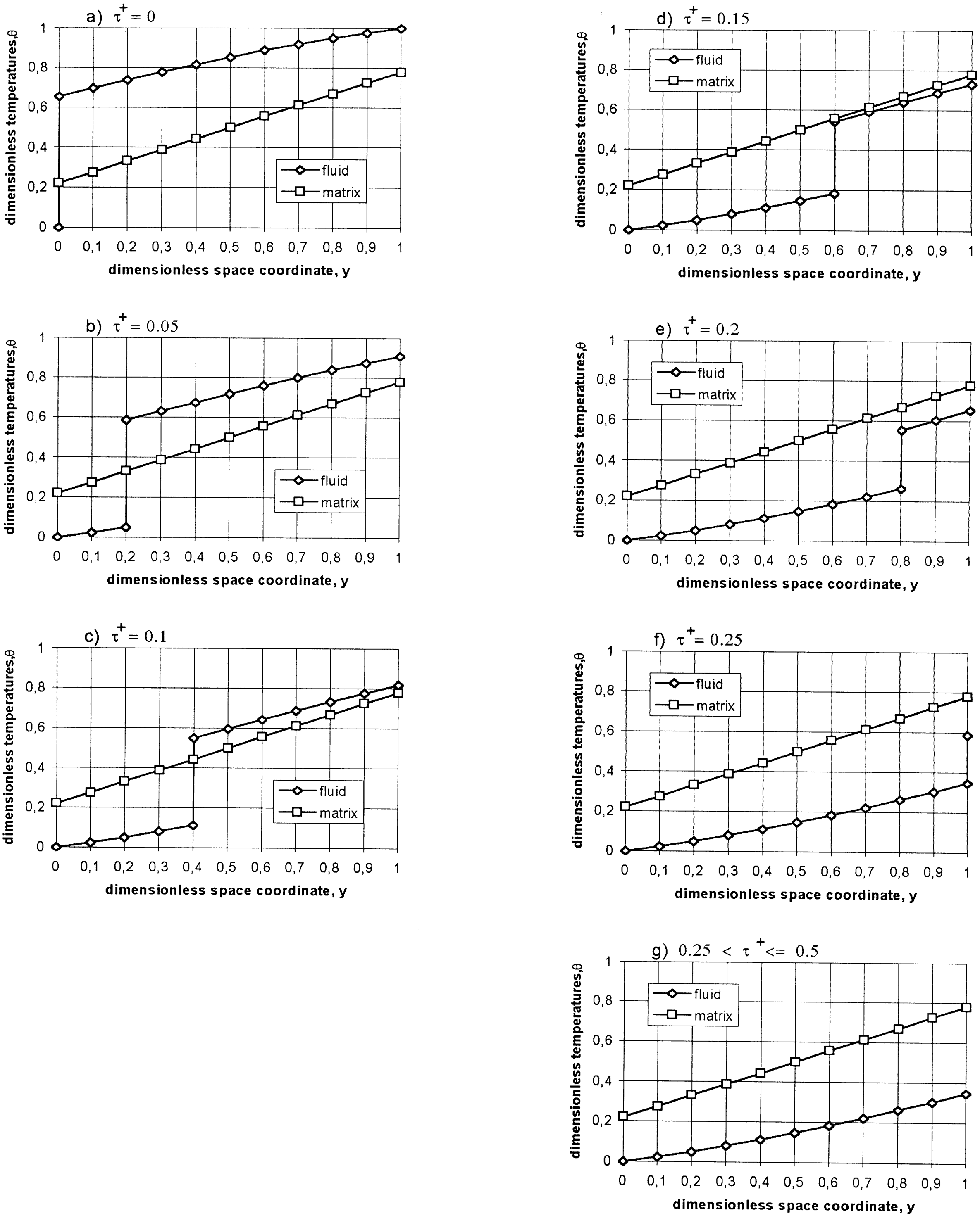

Fig. 3. Dimensionless temperatures (fluid and matrix) as a function of dimensionless length y with τ^+ as a parameter during the blow period. In all case (a)–(g) $\varepsilon = 0.5$ and $\Lambda = 1$.

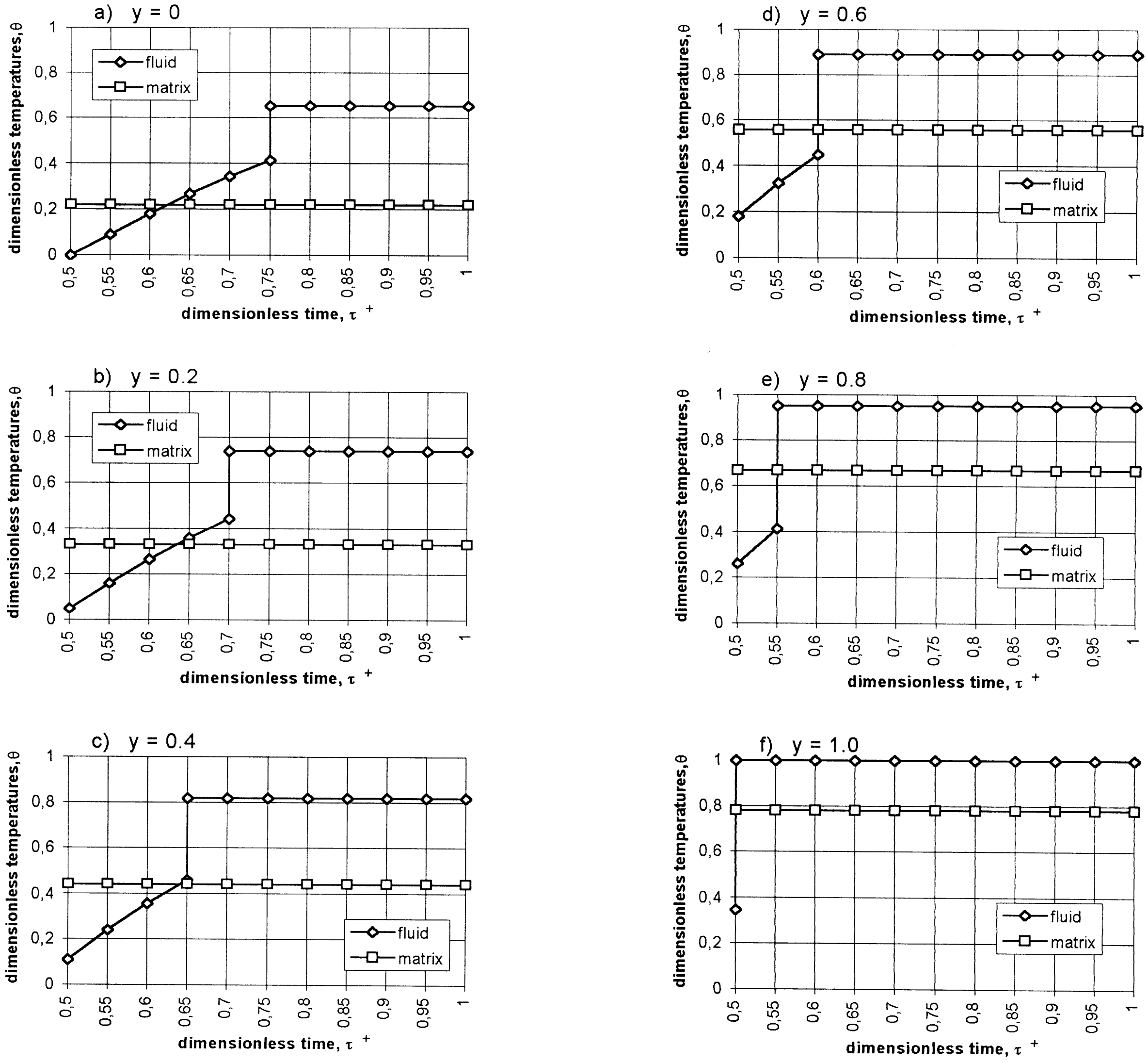

Fig. 4. Dimensionless temperatures (fluid and matrix) as a function of dimensionless time τ^+ with y as a parameter during the reverse period. In all cases (a)–(f) $\varepsilon = 0.5$ and $\Lambda = 1$.

regenerator in counterflow to be analysed. In particular, it allows the effects of the flushing phase on the thermal response of the regenerator to be established. They are:

- the matrix temperature is not a linear function of the location;
- the fluid temperature depends on the time although the thermal capacity of the matrix has been assumed infinitely large. In particular, this time-dependence happens in the first part both of the blow period and of the reverse period;
- when the blow period (known as the heating period for the fluid) starts, the fluid gives up heat to the matrix and only subsequently absorbs heat from the matrix;
- when the reverse period (known as the cooling period for the fluid) starts, the fluid absorbs heat from the matrix and only subsequently supplies heat to the matrix.

In addition, the area enclosed between the dimensionless temperature curves of fluid and matrix at a given location within the regenerator during a semi-period represents the dimensionless heat exchanged between matrix and gas at that location during that semi-period. Thus, a ready estimation of this heat may be obtained graphically.

The obtained results are very useful for rapidly cycled heat regenerators adopted in Stirling cycle machines and gas turbines, where the phenomenon of the flushing phase happens and the thermal capacity of the gas during a

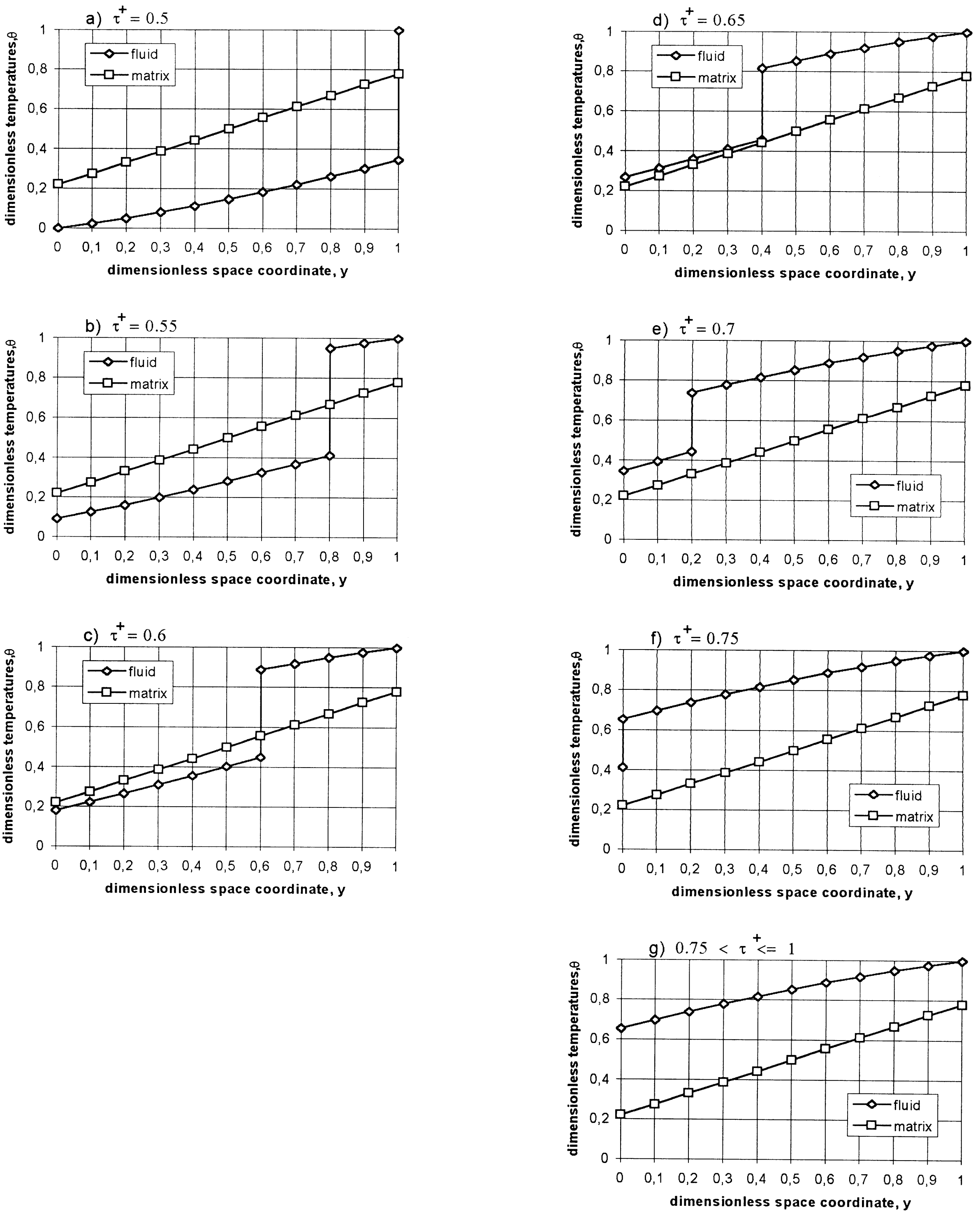

Fig. 5. Dimensionless temperatures (fluid and matrix) as a function of dimensionless length y with τ^+ as a parameter during the reverse period. In all cases (a)–(g) $\varepsilon = 0.5$ and $\Lambda = 1$.

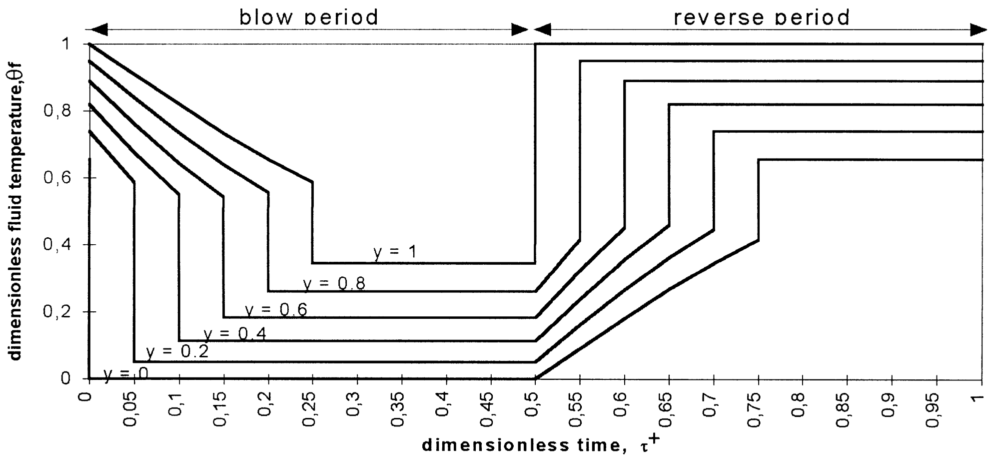

Fig. 6. Dimensionless fluid temperatures as a function of τ^+ with y as a parameter during the whole regenerative period $\tau^+ \in [0, 1]$.

blow is much smaller than the thermal capacity of the matrix.

Appendix A

The functions $H(y)$ and $K(s, y)$ appearing in equation (46) are

$$H(y) = \frac{\varepsilon}{\Lambda(\Lambda+2)}[e^{-\Lambda(1-y)} - e^{-\Lambda y}]$$

$$K(s, y) = \begin{cases} K_1(s, y) & y \in [0, y] \\ K_2(s, y) & y \in [y, 1] \end{cases}$$

$$K_1(s, y) = \varepsilon[e^{\Lambda(s-y)}(\Lambda/\varepsilon - \Lambda y + 1 + \Lambda s) - G(s, y)]/2$$

$$K_2(s, y) = \varepsilon[e^{-\Lambda(s-y)}(\Lambda/\varepsilon + \Lambda y + 1 - \Lambda s) - G(s, y)]/2$$

$$G(s, y) = [e^{-\Lambda(s+y)} + e^{\Lambda(s+y-2)}]/2.$$

The function $K(s, y)$ is generally known as the kernel of the integral equation.

Appendix B

The constants A_k, B_k and C_k $(k = 1, 2)$ appearing in equation (47) are

$$A_1 = A_{1,1}/(\Lambda+2)$$

$$B_1 = A_{1,2}/(\Lambda+2)$$

$$C_1 = (A_{1,3} + A_0)/(\Lambda+2)$$

$$A_2 = B_{1,1}/(\Lambda+2)$$

$$B_2 = B_{1,2}/(\Lambda+2)$$

$$C_2 = (B_{1,3} + B_0)/(\Lambda+2)$$

where the coefficients A_0, B_0, $A_{1,k}$ and $B_{1,k}$ $(k = 1, 2, 3)$ have the following expressions:

$$A_0 = \varepsilon\, e^{-\Lambda}/\Lambda$$

$$B_0 = -\varepsilon/\Lambda$$

$$A_{1,1} = -\varepsilon^2 e^{-\Lambda}/4$$

$$B_{1,1} = \varepsilon^2/4$$

$$A_{1,2} = \frac{\varepsilon}{\Lambda} e^{-\Lambda}\left[\frac{\Lambda}{2}(\varepsilon - 1) + \frac{\varepsilon}{2}(e^{-\Lambda}/2 - 1)\right]$$

$$B_{1,2} = \frac{\varepsilon}{\Lambda}\left[\frac{\varepsilon}{2}(e^{-\Lambda}/2 - 1) - \Lambda/2\right]$$

$$A_{1,3} = \frac{\varepsilon}{2\Lambda} e^{-\Lambda}\left[2\Lambda\left(1 - \frac{\varepsilon}{2}\right) + \left(\varepsilon + \frac{1}{2}\right) + \frac{\varepsilon}{4\Lambda} e^{-2\Lambda} + \frac{e^{-\Lambda}}{2}\left(1 + \frac{\varepsilon}{2\Lambda}\right)\right]$$

$$B_{1,3} = -\frac{\varepsilon}{\Lambda}\left[\frac{e^{-\Lambda}}{4}(1+\varepsilon) + \frac{\varepsilon}{8\Lambda}(1 + e^{-\Lambda})\, e^{-\Lambda} + 1/4\right].$$

Appendix C

The functions $\pi_k(y)$ $(k = 1, 2, 3)$ and the constants σ_k and μ_k $(k = 1, 2, 3)$, which appear in equations (48)–(51), are

$$\pi_1(y) = \alpha_1(y)\Lambda + \beta_1(y)\, e^{2\Lambda y} + \gamma_1$$

$$\pi_2(y) = \alpha_1(y)\Lambda + \beta_1(y)\, e^{2\Lambda y} + \gamma_2$$

$$\pi_3(y) = \alpha_2(y)\Lambda - \beta_2(y)\, e^{-2\Lambda y} + \gamma_1$$

$$\sigma_1 = -A_2\Lambda/3$$

$$\sigma_2 = (A_2 - B_2\Lambda)/2$$

$$\sigma_3 = (A_2\Lambda + B_2)/2 - C_2\Lambda$$

$$\mu_1 = -A_1\Lambda/3$$

$$\mu_2 = -(B_1\Lambda + A_1)/2$$

$$\mu_3 = (A_1\Lambda - B_1)/2 - C_1\Lambda$$

where

$$\alpha_1(y) = \frac{A_2}{3}y^3 + \frac{B_2}{2}y^2 + C_2 y$$

$$\beta_1(y) = \frac{1}{2}\left[A_1 y^2 + \left(B_1 - \frac{A_1}{\Lambda}\right)y - 2\gamma_1\right]$$

$$\gamma_1 = \frac{1}{2}\left[\left(B_1 - \frac{A_1}{\Lambda}\right)\frac{1}{2\Lambda} - C_1\right]$$

$$\alpha_2(y) = \frac{A_1}{3}y^3 + \frac{B_1}{2}y^2 + C_1 y$$

$$\beta_2(y) = \frac{1}{2}\left[A_2 y^2 + \left(B_2 + \frac{A_2}{\Lambda}\right)y + 2\gamma_2\right]$$

$$\gamma_2 = \frac{1}{2}\left[\left(B_2 + \frac{A_2}{\Lambda}\right)\frac{1}{2\Lambda} + C_2\right]$$

and the constants A_k, B_k and C_k ($k = 1, 2$) are given in Appendix B.

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